

## ON CHEREDNIK-MACDONALD-MEHTA IDENTITIES

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## INTRODUCTION

In this note we give a proof of Cherednik's generalization of Macdonald–Mehta identities for the root system  $A_{n-1}$  using the representation theory of quantum groups. These identities, suggested and proved in [Ch2], give an explicit formula for the integral of a product of Macdonald polynomials with respect to a “difference analogue of the Gaussian measure”. They can be written for any reduced root system, or, equivalently, for any semisimple complex Lie algebra  $\mathfrak{g}$ . Assuming for simplicity that  $\mathfrak{g}$  is simple and simply-laced, these identities are given by the following formula:

$$(1) \quad \frac{1}{|W|} \int \delta_k \overline{\delta_k} P_\lambda \overline{P_\mu} \gamma \, dx = q^{\lambda^2 + (\mu, \mu + 2k\rho)} P_\mu(q^{-2(\lambda + k\rho)}) \\ \times q^{-2k(k-1)|R_+|} \prod_{\alpha \in R_+} \prod_{i=0}^{k-1} (1 - q^{2(\alpha, \lambda + k\rho) + 2i})$$

where  $\lambda, \mu$  are dominant integral weights,  $k$  is a positive integer,  $P_\lambda$  are Macdonald polynomials associated with the corresponding root system, with parameters  $q^2, t = q^{2k}$  (see [M1, M2] or a review in [K2]),  $\delta_k$  is the  $q$ -analogue of  $k$ -th power of the Weyl denominator  $\delta = \delta_1$ :

$$(2) \quad \delta_k = \prod_{\alpha \in R^+} \prod_{i=0}^{k-1} (e^{\alpha/2} - q^{-2i} e^{-\alpha/2}),$$

and  $\gamma$  is the Gaussian, which we define by

$$(3) \quad \gamma = \sum_{\lambda \in P} e^\lambda q^{\lambda^2},$$

where  $P$  is the weight lattice. We consider  $\gamma$  as a formal series in  $q$  with coefficients from the group algebra of the weight lattice. In a more standard terminology  $\gamma$  is called the theta-function of the lattice  $P$ . All other notations, which are more or less standard, will be explained below.

These identities were formulated in the form we use in a paper of Cherednik [Ch2], who also proved them using double affine Hecke algebras (note: our notations are somewhat different from Cherednik's ones). We refer the reader to [Ch2] for the discussion of the history of these identities and their role in difference harmonic analysis.

As an important intermediate step, we also prove the following identity for the Gaussian:

$$(4) \quad \gamma = \left( \prod_{\alpha \in R_+} (1 - q^{2(\alpha, \rho)}) \right) \sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_q L_\nu) \chi_\nu.$$

Here  $\chi_\nu$  is the character of the irreducible finite-dimensional module  $L_\nu$  over  $\mathfrak{g}$ , and  $\dim_q L_\nu := \chi_\nu(q^{2\rho})$  is the quantum dimension of  $L_\nu$ . This identity was known to experts and is not difficult to prove; however, we were unable to locate a proof in the literature.

**Notations.** We use the same notations as in [EK1, EK2] with the following exceptions: we replace  $q$  by  $q^{-1}$  (note that this does not change the Macdonald's polynomials) and we use the notation  $\varphi_\lambda$  for “generalized characters” (see below), reserving the notation  $\chi_\lambda$  for usual (Weyl) characters. In particular, we define  $\overline{e^\lambda} = e^{-\lambda}$ ,  $\bar{q} = q$ , and for  $f \in \mathbb{C}_q[P]$ , we define  $f(q^\lambda)$ ,  $\lambda \in P$  by  $e^\mu(q^\lambda) = q^{(\mu, \lambda)}$ . For brevity, we also write  $\lambda^2$  for  $(\lambda, \lambda)$ . Finally, we denote by  $\int dx : \mathbb{C}_q[P] \rightarrow \mathbb{C}_q$  the functional of taking the constant term:  $\int e^\lambda dx = \delta_{\lambda, 0}$ .

## 1. REWRITING THE GAUSSIAN

In this Section, we prove formula (4) for an arbitrary simple Lie algebra  $\mathfrak{g}$ .

**Proposition 1.** *Let  $\gamma$  be defined by (3). Then*

$$\gamma = \left( \prod_{\alpha \in R_+} (1 - q^{2(\alpha, \rho)}) \right) \sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_q L_\nu) \chi_\nu.$$

*Proof.* The proof is straightforward and uses Weyl character formula along with the following result: if we extend the definition of  $\chi_\nu$  to all  $\nu \in P$  by letting  $\chi_\nu = (\sum (-1)^{|w|} e^{w(\nu + \rho)}) / \delta$  (recall that  $\delta = \delta_1$  is the Weyl denominator) then  $\chi_{w.\nu} = (-1)^{|w|} \chi_\nu$ , where  $w.\nu = w(\nu + \rho) - \rho$ . In particular,  $\chi_\nu = 0$  if  $\nu$  lies on one of the walls, i.e. if  $s_\alpha.\nu = \nu$  for some root  $\alpha$ . The same applies to  $\dim_q L_\nu = \chi_\nu(q^{2\rho})$ . Using this, we rewrite the right-hand side of (4) as follows:

$$\begin{aligned} \sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_q L_\nu) \chi_\nu &= q^{-\rho^2} \sum_{\nu \in P_+} q^{(\nu + \rho)^2} (\dim_q L_\nu) \chi_\nu \\ &= \frac{q^{-\rho^2}}{|W|} \sum_{\nu \in P} q^{\nu^2} (\dim_q L_{\nu - \rho}) \chi_{\nu - \rho}. \end{aligned}$$

By Weyl character formula, we can write

$$(\dim_q L_{\nu - \rho}) \chi_{\nu - \rho} = \frac{1}{\delta(q^{2\rho})} \frac{1}{\delta} \sum (-1)^{|w_1 w_2|} q^{2(\nu, w_1(\rho))} e^{w_2(\nu)}.$$

Substituting this in the previous identity, we get

$$\begin{aligned}
\sum_{\nu \in P_+} q^{(\nu, \nu+2\rho)} (\dim_q L_\nu) \chi_\nu &= \frac{q^{-2\rho^2}}{|W|\delta(q^{2\rho})} \frac{1}{\delta} \sum_{\substack{w_1, w_2 \in W \\ \nu \in P}} q^{(\nu+w_1(\rho))^2} (-1)^{|w_1 w_2|} e^{w_2(\nu)} \\
&= \frac{q^{-2\rho^2}}{|W|\delta(q^{2\rho})} \frac{1}{\delta} \sum_{\substack{w_1, w_2 \in W \\ \lambda \in P}} q^{\lambda^2} (-1)^{|w_1 w_2|} e^{w_2(\lambda-w_1(\rho))} \\
&= \frac{q^{-2\rho^2} (-1)^{|R_+|}}{|W|\delta(q^{2\rho})} \sum_{\substack{w_2 \in W \\ \lambda \in P}} q^{\lambda^2} e^{w_2(\lambda)} = \frac{q^{-2\rho^2} (-1)^{|R_+|}}{\delta(q^{2\rho})} \sum_{\lambda \in P} q^{\lambda^2} e^\lambda
\end{aligned}$$

Simplifying this, we get the statement of the Proposition.  $\square$

**Example.** Let  $\mathfrak{g} = \mathfrak{sl}_2$ . Then Proposition 1 gives the following identity, which can be verified directly:

$$(5) \quad \sum_{n \geq 0} q^{n(n+2)/2} [n+1] (x^n + x^{n-2} + \dots + x^{-n}) = \frac{1}{1-q^2} \sum_{l \in \mathbb{Z}} x^l q^{l^2/2},$$

where  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ .

Finally, we note that the Gaussian can be defined in any semisimple ribbon category  $\mathcal{C}$ , i.e. a tensor category, with possibly non-trivial commutativity isomorphism, and a “Casimir element” (also called “ribbon element”) satisfying certain compatibility properties (see, e.g., [Kas] or [K1]). Namely, we define the Gaussian to be the following element of the suitable completion of the Grothendieck ring  $K(\mathcal{C})$ :

$$\gamma_{\mathcal{C}} = \sum_i C_i \dim X_i \langle X_i \rangle,$$

where  $X_i$  are simple objects in  $\mathcal{C}$ ,  $C_i$  is the value of the Casimir element in  $X_i$  (in [K1], these numbers are denoted by  $\theta_i$ ),  $\dim X_i$  is the  $q$ -dimension of  $X_i$ , and  $\langle X_i \rangle$  is the class of  $X_i$  in the Grothendieck ring. In particular, if we take the category of representations of the quantum group  $U_q \mathfrak{h}$  corresponding to the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  considered as a commutative Lie algebra, then its irreducible representations are parametrized by  $\lambda \in P$ , and they are all one-dimensional. One can check that defining the universal  $R$ -matrix by  $R|_{X_\lambda \otimes X_\mu} = q^{(\lambda, \mu)}$ , and the Casimir element by  $C|_{X_\lambda} = q^{\lambda^2}$  endows this category with a structure of ribbon category. Thus, the Gaussian  $\gamma_{U_q \mathfrak{h}} = \gamma$  for this category is exactly given by the formula (3). On the other hand, if we consider the category of representations of the quantum group  $U_q \mathfrak{g}$ , then the Casimir element  $C$  in this category is defined by  $C = q^{2\rho} u^{-1}$ , where  $u$  is the Drinfeld’s element (see details in [Kas, Chapter XIV.6], where the element  $\theta = C^{-1}$  is discussed), and  $C|_{L_\lambda} = q^{(\lambda, \lambda+2\rho)}$ . Thus, Gaussian for this category is given by  $\gamma_{U_q \mathfrak{g}} = \sum_{\nu \in P_+} q^{(\nu, \nu+2\rho)} (\dim_q L_\nu) \chi_\nu$ . Therefore, Proposition 1 can be rewritten as

$$\gamma_{U_q \mathfrak{g}} = \left( \prod_{\alpha \in R_+} \frac{1}{1 - q^{2(\alpha, \rho)}} \right) \gamma_{U_q \mathfrak{h}},$$

which is closely connected with the Weyl formula for a compact group  $G$ , which relates the measure on  $G/\text{Ad } G = T/W$  coming from the Haar measure on  $G$  with the Haar measure on  $T$ .

## 2. PROOF OF CHEREDNIK–MACDONALD–MEHTA IDENTITIES

In this section, we give a proof of the Cherednik–Macdonald–Mehta identities (1) for  $\mathfrak{g} = \mathfrak{sl}_n$ . The proof is based on the realization of Macdonald’s polynomials as “vector-valued characters” for the quantum group  $U_q \mathfrak{sl}_n$ , which was given in [EK1]. For the sake of completeness, we briefly outline these results here, referring the reader to the original paper for a detailed exposition.

Let us fix  $k \in \mathbb{Z}_+$  and denote by  $U$  the finite-dimensional representation of  $U_q \mathfrak{sl}_n$  with highest weight  $n(k-1)\omega_1$ , where  $\omega_1$  is the first fundamental weight. We identify the zero weight subspace  $U[0]$ , which is one-dimensional, with  $\mathbb{C}_q$ .

For  $\lambda \in P_+$ , we denote by  $\Phi_\lambda$  the unique intertwiner

$$\Phi_\lambda : L_{\lambda+(k-1)\rho} \rightarrow L_{\lambda+(k-1)\rho} \otimes U$$

and define the generalized character  $\varphi_\lambda \in \mathbb{C}_q[P] \otimes U[0] \simeq \mathbb{C}_q[P]$  by  $\varphi_\lambda(q^x) = \text{Tr}_{L_{\lambda+(k-1)\rho}}(\Phi_\lambda q^x)$ .

We can now summarize the results of [EK1] as follows:

$$(6) \quad \varphi_0 = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} (e^{\alpha/2} - q^{-2i} e^{-\alpha/2}) = \delta_k / \delta$$

$$\varphi_\lambda / \varphi_0 = P_\lambda$$

where  $P_\lambda$  is the Macdonald polynomial with parameters  $q^2, t = q^{2k}$ .

We can also rewrite Macdonald’s inner product in terms of the generalized characters as follows. Recall that Macdonald’s inner product on  $\mathbb{C}_q[P]$  is defined by

$$\langle f, g \rangle_k = \frac{1}{|W|} \int \delta_k \bar{\delta}_k f \bar{g} dx$$

(this differs by a certain power of  $q$  from the original definition of Macdonald). Obviously, one has

$$\langle P_\lambda, P_\mu \rangle_k = \langle \varphi_\lambda, \varphi_\mu \rangle_1.$$

In order to rewrite this in terms of representation theory, let  $\omega$  be the Cartan involution in  $U_q \mathfrak{sl}_n$  (see [EK1]). For a  $U_q \mathfrak{sl}_n$ -module  $V$ , we denote by  $V^\omega$  the same vector space but with the action of  $U_q \mathfrak{sl}_n$  twisted by  $\omega$ . Note that for finite-dimensional  $V$ , we have  $V^\omega \simeq V^*$  (not canonically). Similarly, for an intertwiner  $\Phi : L \rightarrow L \otimes U$  we denote by  $\Phi^\omega$  the corresponding intertwiner  $L^\omega \rightarrow U^\omega \otimes L^\omega$ . Finally, for  $\Phi_1 : L_1 \rightarrow L_1 \otimes U, \Phi_2 : L_2 \rightarrow L_2 \otimes U$ , define  $\Phi_1 \odot \Phi_2 \in \text{End}(L_1 \otimes L_2^\omega)$  as the composition  $L_1 \otimes L_2^\omega \rightarrow L_1 \otimes U \otimes U^\omega \otimes L_2^\omega \rightarrow L_1 \otimes L_2^\omega$ , where the first arrow is given by  $\Phi_1 \otimes \Phi_2^\omega$ , and the second by the invariant pairing  $U \otimes U^\omega \rightarrow \mathbb{C}_q$  (which is unique up to a constant). Then it was shown in [EK1] that

$$(\varphi_\lambda \overline{\varphi_\mu})(q^x) = \text{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega) q^{\Delta(x)}) = \sum_{\nu \in P_+} \chi_\nu(q^x) C_{\lambda\mu}^\nu$$

where  $V = L_{\lambda+(k-1)\rho} \otimes L_{\mu+(k-1)\rho}^\omega$  and  $C_{\lambda\mu}^\nu$  is the trace of  $\Phi_\lambda \odot \Phi_\mu$  acting in the multiplicity space  $\text{Hom}(L_\nu, V)$ . As a corollary, we get the following result:

$$(7) \quad \frac{1}{|W|} \int \delta \bar{\delta} \varphi_\lambda \overline{\varphi_\mu} \left( \sum a_\nu \chi_\nu \right) dx = \sum a_{\nu^*} C_{\lambda\mu}^\nu,$$

where  $\nu^* = -w_0(\nu)$  is the highest weight of the module  $(L_\nu)^*$  (here  $w_0$  is the longest element of the Weyl group).

Of course, the coefficients  $C_{\lambda\mu}^\nu$  are very difficult to calculate. However, the formula above is still useful. For example, it immediately shows that  $\langle \varphi_\lambda, \varphi_\mu \rangle_1 = 0$  unless  $\lambda = \mu$ , which was the major part of the proof of the formula  $\varphi_\lambda/\varphi_0 = P_\lambda$  in [EK1]. It turns out that this formula also allows us to prove the Cherednik-Macdonald-Mehta identities.

**Theorem 2.** *Let  $\varphi_\lambda$  be the renormalized Macdonald polynomials for the root system  $A_{n-1}$  given by (6), and let  $\gamma$  be the Gaussian (3). Then*

$$(8) \quad \frac{1}{|W|} \int \delta \bar{\delta} \varphi_\lambda \overline{\varphi_\mu} \gamma dx = q^{(\lambda+k\rho)^2} q^{(\mu+k\rho)^2} \varphi_\mu(q^{-2(\lambda+k\rho)}) \\ \times \left( \prod_{\alpha \in R_+} (1 - q^{2(\alpha, \rho)}) \right) q^{-2\rho^2} \|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho},$$

where  $\|P_\lambda\|^2 = \langle P_\lambda, P_\lambda \rangle_k$ .

*Proof.* From (7) and (4), we get

$$(9) \quad \int \delta \bar{\delta} \varphi_\lambda \overline{\varphi_\mu} \gamma dx = \left( \prod_{\alpha \in R_+} (1 - q^{2(\alpha, \rho)}) \right) \sum_{\nu \in P^+} q^{(\nu, \nu+2\rho)} (\dim_q L_\nu) C_{\lambda\mu}^\nu.$$

On the other hand, let  $C$  be the Casimir element for  $U_q \mathfrak{g}$  discussed above. Consider the intertwiner  $(\Phi_\lambda \odot \Phi_\mu^\omega) \Delta(C) : V \rightarrow V$ , where, as before,  $V = L_{\lambda+(k-1)\rho} \otimes L_{\mu+(k-1)\rho}^\omega$ . Then it follows from  $C|_{L_\lambda} = q^{(\lambda, \lambda+2\rho)}$  that

$$\mathrm{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega) \Delta(C) \Delta(q^{2\rho})) = \sum_{\nu \in P_+} C_{\lambda\mu}^\nu q^{(\nu, \nu+2\rho)} \dim_q L_\nu$$

which is exactly the sum in the right hand side of (9). On the other hand, using  $\Delta(C) = (C \otimes C)(R^{21}R)$ , we can write

$$\mathrm{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega) \Delta(C) \Delta(q^{2\rho})) = q^{-2\rho^2} q^{(\lambda+k\rho)^2} q^{(\mu+k\rho)^2} \mathrm{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega)(R^{21}R) \Delta(q^{2\rho}))$$

This last trace can be calculated, which was done in [EK2, Corollary 4.2], and the answer is given by

$$\mathrm{Tr}_V((\Phi_\lambda \odot \Phi_\mu^\omega)(R^{21}R) \Delta(q^{2\rho})) = \varphi_\mu(q^{-2(\lambda+k\rho)}) \|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho}.$$

Combining these results, we get the statement of the theorem.  $\square$

The norms  $\|P_\lambda\|^2$  appearing in the right-hand side of (8) are given by Macdonald's inner product identities

$$\|P_\lambda\|^2 = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} \frac{1 - q^{-2(\alpha, \lambda+k\rho)-2i}}{1 - q^{-2(\alpha, \lambda+k\rho)+2i}},$$

which were conjectured in [M1, M2] and proved for root system  $A_{n-1}$  by Macdonald himself [M2]; see also [EK2] for the proof based on representation theory of  $U_q$ .

and [Ch1] or a review in [K2] for a proof for arbitrary root systems. Using this formula and rewriting the statement of Theorem 2 in terms of Macdonald polynomials  $P_\lambda$  rather than  $\varphi_\lambda$ , we get the Cherednik–Macdonald–Mehta identities (1).

*Remarks.* 1. Note that the left-hand side of (8) is symmetric in  $\lambda, \mu$ . Thus, the same is true for the right-hand side, which is exactly the statement of Macdonald’s symmetry identity (compare with the proof in [EK2]).

2. The proof of Cherednik–Macdonald–Mehta identities given above easily generalizes to the case when  $q$  is a root of unity (see [K1] for the discussion of the appropriate representation-theoretic setup). In this case, we need to replace the set  $P_+$  of all integral dominant weights by an appropriate (finite) Weyl alcove  $C$  (see [K1]), and the integral  $\int \delta \bar{\delta} f dx$  should be replaced by  $\text{const} \sum_{\lambda \in C} f(q^{2(\lambda+\rho)}) \dim_q L_\lambda$ . Using the following obvious property of the Gaussian:

$$\gamma(q^{2(\lambda+\rho)}) = q^{-(\lambda, \lambda+2\rho)} \gamma(q^{2\rho})$$

(which in this case coincides with formula (1.7) in [K1]), it is easy to see that in this case the Cherednik–Macdonald–Mehta identities are equivalent to

$$S^{-1}T^{-1}S = TST$$

where the matrices  $S, T$  are defined in [K1, Theorem 5.4]. This identity is a part of a more general result, namely, that these matrices  $S, T$  give a projective representation of the modular group  $SL_2(\mathbb{Z})$  on the space of generalized characters (see [K1, Theorem 1.10] and references therein).

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